On the 90-degree-lemma

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Abstract

In their technical report "On the bitopological nature of Stone duality" Jung and Moshier axiomatise a bitopological space as a *d-frame*, which can equivalently be described as a *partial frame*, a structure with two orders, one being a special Scott domain and the other a complete lattice. The rich interaction of these two orders arises from a ternary operation on distributive lattices and is informally known as the *90-degree-lemma*. Motivation for considering a second order originates in Belnap's four-valued logic. The infinitary connections of the two orders are based on a set of axioms which are derived from the Stone duality for bitopological spaces. In this paper it is shown that the axioms given by Jung and Moshier contain some previously unknown redundancies. The redundancies yield an isomorphism of two categories, one having special Scott domains as objects and the other a certain type of complete lattice.

Introduction

The 90-degree-lemma is a purely order-theoretic fact about bounded distributive lattices. Nevertheless it proves to be particularly useful when working with Belnap's four-valued logic [1]. The set of truth values **4** Belnap uses can be equipped with two different orders. One of them can be thought of as the logical order and the other as measuring informational content. Accordingly the top element in the information order, usually denoted as $\top$ stands for *too much* or *contradicting* information, whereas the least element $\perp$ means *no information*. The classical boolean truth values "true" and "false" will be denoted by t and f. The four values can be presented as subsets of the classical set of truth values by $\perp = \{\}$, $t = \{\text{true}\}$, $f = \{\text{false}\}$, $\top = \{\text{false}, \text{true}\}$. The figure below illustrates

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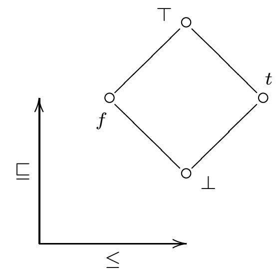
why the relation between the two orders is described by “90 degrees”.



One can see that by rotating the diagram by 90 degrees[1] we obtain the Hasse diagram for the other order, and this operation is reversible. This fact has been coined *the 90-degree-lemma* by A. Jung and M. A. Moshier [6]. In the following we study the translation operation more closely.

1 The 90-degree-lemma

The basis for the 90-degree-lemma is a ternary operation on distributive lattices, which has been studied for at least 60 years. G. Birkhoff and S. A. Kiss [2] define

$$(a, b, c) = (a \wedge b) \vee (b \wedge c) \vee (c \wedge a). \tag{1.1}$$

By a simple transformation using the distributive law one finds this to be equal to

$$(a \vee b) \wedge (b \vee c) \wedge (c \vee a). \tag{1.2}$$

This is only one of the many symmetries this term possesses. We shall use a notation that fits our intuition better and write $a \underset{b}{\sqcap} c = (a, b, c)$. Other symmetries of the ternary operation (1.1), collected from works by G. Birkhoff [2] and A. A. Grau [4], are

$$a \underset{b}{\sqcap} c = b \underset{a}{\sqcap} c = b \underset{c}{\sqcap} a, \tag{1.3}$$

$$a \underset{b}{\sqcap} a = a, \tag{1.4}$$

$$a \underset{b}{\sqcap} b = b \underset{b}{\sqcap} a = b, \tag{1.5}$$

$$(a \underset{b}{\sqcap} c) \underset{d}{\sqcap} e = (a \underset{d}{\sqcap} e) \underset{b}{\sqcap} (c \underset{d}{\sqcap} e), \tag{1.6}$$

$$(a \underset{b}{\sqcap} c) \underset{b}{\sqcap} a = a \underset{b}{\sqcap} c, \tag{1.7}$$

$$(a \underset{b}{\sqcap} c) \underset{b}{\sqcap} d = a \underset{b}{\sqcap} (c \underset{b}{\sqcap} d) \tag{1.8}$$

[1] more precisely reflecting at the diagonal

Further, where the complementation operation $'$ in a bounded distributive lattice is defined[2] we have

$$a \underset{b}{\sqcap} b' = b' \underset{b}{\sqcap} a = a, \tag{1.9}$$

$$a \underset{b}{\sqcap} a' = b, \tag{1.10}$$

$$a \underset{b}{\sqcap} c = (a \underset{x}{\sqcap} b) \underset{x'}{\sqcap} (b \underset{x}{\sqcap} c) \underset{x'}{\sqcap} (c \underset{x}{\sqcap} a) \tag{1.11}$$

The following is the 90-degree-lemma as stated in [6, Prop. 3.2].

Proposition 1.1. *[90-degree-lemma] Let $(L, \vee, \wedge, f, t)$ be a bounded distributive lattice and $\langle \top, \bot \rangle$ a complemented pair, that is $\top \vee \bot = t$ and $\top \wedge \bot = f$. Then by the operations*

$$x \sqcap y \quad := \quad x \underset{\bot}{\sqcap} y = (x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y) \tag{1.12}$$

$$x \sqcup y \quad := \quad x \underset{\top}{\sqcap} y = (x \wedge \top) \vee (y \wedge \top) \vee (x \wedge y) \tag{1.13}$$

one obtains another bounded distributive lattice $(L, \sqcap, \sqcup, \bot, \top)$ in which $\langle f, t \rangle$ is a complemented pair. Moreover, the operations $\vee, \wedge, \sqcap, \sqcup$ distribute over each other. By substituting f for $\bot$, t for $\top$, $\sqcap$ for $\vee$ and $\sqcup$ for $\vee$ in (1.12) and (1.13) one one can recover the original lattice operations.

We will refer to the order $\sqsubseteq$ induced by $\sqcap$ as the *information order* and to $\leq$ induced by $\wedge$ as the *logical order* on L.

The proof of this can be easily obtained from the symmetries and identities of the ternary operation. The operation $\sqcap$ is associative by (1.8), commutative by (1.3), idempotent because of (1.4) and $\bot$ is the least element by (1.5). The absorbtive laws are derived from (1.6) and (1.11). The distributive law for $\sqcap$ and $\sqcup$ is exactly (1.6). Also by (1.6) one can see that $\wedge, \vee, \sqcap$ and $\sqcup$ distribute over one another. Complemented pairs are mutually preserved by (1.10).

G. Birkhoff mentions the 90-degree-lemma as presented above in his paper in the following form.
Define a pair $\langle a, b \rangle$ in a distributive lattice L to be complementary *if*

$$\forall x \in L.\, a \underset{x}{\sqcap} b = x. \tag{1.14}$$

Then $x \sqcap y = x \underset{a}{\sqcap} y$, $x \sqcup y = x \underset{b}{\sqcap} y$ yields a distributive lattice structure on the lattice L. What Birkhoff did not mention is that his notion of a complementary pair $\langle a, b \rangle$ is actually equivalent to the lattice L being bounded and $\langle a, b \rangle$ being a complemented pair therein.

Indeed, it is readily seen that if $0, 1$ are least and greatest elements of L, then (1.14) implies that $(a, 0, b) = a \wedge b = 0$ and $(a, 1, b) = a \vee b = 1$ which

[2] Grau proves these identities for the ternary operation on boolean algebras, but they remain true in a bounded distributive lattice.

means that $\langle a, b\rangle$ is complemented in the classical sense. Conversely, suppose $\langle a, b\rangle$ satisfies (1.14). Then for all $\bot \leq a \wedge b$ and $\top \geq a \vee b$ we have

$$\begin{aligned} (a, \bot, b) &= ((a \vee b) \wedge \bot) \vee (a \wedge b) \\ &= a \wedge a = \bot \\ (a, \top, b) &= ((a \vee b) \wedge \top) \vee (a \wedge b) \\ &= a \vee b = \top \end{aligned}$$

which shows that $a \vee b$ is a maximal element in L and $a \wedge b$ is a minimal element in L. Now in a lattice maximal and minimal elements are unique, so L is bounded by $a \wedge b$ and $a \vee b$ and trivially $\langle a, b\rangle$ is complemented in the classical sense.

Grau shows in his work [4] that on a boolean algebra the function

$$f(x) = (b' \underset{a}{\sqcap} x) \underset{a'}{\sqcap} (b \underset{a}{\sqcap} x')$$

is a self-inverse automorphism of the boolean algebra which transforms a into b, that is

$$\begin{aligned} f(a \underset{x}{\sqcap} b) &= f(a) \underset{f(x)}{\sqcap} f(b), \\ f(a) &= b, \\ f^2 &= \mathrm{id}\,. \end{aligned}$$

Moreover, on a boolean algebra the ternary operation (1.1) is the only way to realise a ternary boolean algebra structure in the sense of Grau.

The following fact can be thought of as a generalisation of Grau's work, and appears to be new.

Lemma 1.2. *Let L be a set and $(\wedge, \vee)$ and $(\sqcap, \sqcup)$ be two bounded distributive lattice structures on L which distribute over each other. If $\bot$ is the neutral element of $\sqcup$ and $\top$ is the neutral element of $\sqcap$ then equations (1.12) and (1.13) hold. Consequently the neutral elements of one pair of operations is a complemented pair for the other.*

Proof. Suppose $(\wedge, \vee)$ and $(\sqcap, \sqcup)$ distribute over each other. We will only show that

$$x \sqcap y = (x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y)$$

since the equation (1.13) is proven in a dual way.

$$\begin{aligned} & ((x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y)) \sqcap (x \sqcap y) \\ = \; & ((x \wedge \bot) \sqcap (x \sqcap y)) \vee ((y \wedge \bot) \sqcap (x \sqcap y)) \vee ((x \wedge y) \sqcap (x \sqcap y)) \\ = \; & ((x \sqcap y) \wedge \bot) \vee ((x \sqcap y) \wedge \bot) \vee ((x \sqcap y) \wedge (x \sqcap y)) \\ = \; & ((x \sqcap y) \wedge \bot) \vee (x \sqcap y) \\ = \; & x \sqcap y. \end{aligned}$$

Thus we know that $x \sqcap y \sqsubseteq (x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y)$. Further

$$\begin{aligned} (x \wedge y) \wedge (x \sqcap y) &= ((x \wedge y) \wedge x) \sqcap ((x \wedge y) \wedge y) \\ &= (x \wedge y) \sqcap (x \wedge y) \\ &= x \wedge y \\ (x \vee y) \wedge (x \sqcap y) &= (x \wedge (x \vee y)) \sqcap (y \wedge (x \vee y)) \\ &= x \sqcap y \end{aligned}$$

whence we have the inequalities

$$x \wedge y \leq x \sqcap y \leq x \vee y. \tag{1.15}$$

Using this we calculate

$$\begin{aligned} & ((x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y)) \sqcup (x \sqcap y) \\ = \; & (x \wedge (x \sqcap y)) \vee (y \wedge (x \sqcap y)) \vee (x \wedge y) \\ = \; & ((x \sqcap y) \wedge (x \vee y)) \vee (x \wedge y) \\ \overset{(1.15)}{=} \; & x \sqcap y. \end{aligned}$$

Therefore $x \sqcap y \sqsupseteq (x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y)$ which by what is shown above proves (1.12). Note that by swapping the roles of $(\wedge, \vee)$ and $(\sqcap, \sqcup)$ one obtains the corresponding expression of $x \wedge y$ in terms of $\sqcap, \sqcup$ and the neutral element of $\vee$. With this it is easy to show that the neutral elements of $\wedge$ and $\vee$ form a complemented pair in $(L, \sqcap, \sqcup)$ and dually. □

Proposition 1.3. *Let L be a set. Define a relation $\sim$ on the set of all bounded distributive lattice structures on L by saying that $(\wedge, \vee) \sim (\sqcap, \sqcup)$ whenever the four operations distribute over each other. Then $\sim$ is an equivalence relation, and for each lattice structure $(\wedge, \vee)$ there is a bijection between the equivalence class of $(\wedge, \vee)$ and the complemented pairs in the lattice $(L, \wedge, \vee)$. More precisely, if $(\wedge, \vee) \sim (\sqcap, \sqcup)$ then the lattices $(L, \wedge, \vee)$ and $(L, \sqcap, \sqcup)$ have the same set of complemented elements. Moreover, for any two elements a, b in $(L, \wedge, \vee)$ which have a complement, the function*

$$f_{a,b}(x) = (b' \underset{a}{\sqcap} x) \underset{a'}{\sqcap} (b \underset{a}{\sqcap} x')$$

defined on the complemented elements preserves all equivalent lattice operations and maps a to b.

Proof. It is trivial that the relation $\sim$ is reflexive and symmetric. Transitivity of $\sim$ follows from Lemma 1.2 and (1.11) and (1.6). For every complemented pair in $(L, \wedge, \vee)$ the 90-degree-lemma provides us with a bounded distributive lattice structure which distributes over the given operations $\wedge$ and $\vee$. Conversely Lemma 1.2 provides that all such equivalent lattice structures arise by such a complemented pair. By reversibility of (1.12) and (1.13) the correspondence between equivalent lattice structures and complemented pairs is bijective. In any bounded distributive lattice the elements which have a complement form a boolean algebra. The stated properties of $f_{a,b}$ follow from Grau's work. □

2 The 90-degree-lemma without $\top$

In this section we show that it is possible to reconstruct the original lattice from one operation $\cdot \underset{\perp}{\sqcap} \cdot$ alone. We do this in two steps, which also will characterise those semilattices which give rise to a bounded distributive lattice. First we show that a semilattice obtained from a bounded distributive lattice satisfies certain axioms and after that show that any semilattice which satisfies those axioms induces a lattice structure on its set of points in the way of the 90-degree-lemma.

2.1 The semilattice induced by a bounded distributive lattice

Let $(L, \wedge, \vee, f, t)$ be a bounded distributive lattice and $\perp \in L$ any element. We define a semilattice operation $\sqcap$ as in (1.12). This induces the information order on L and we write the lower set of an element in the information order as $\downarrow x$. The following fact can be found in [3, Exercise 6.6, 6.12].

Lemma 2.1. *Let $(L, \wedge, \vee)$ be a distributive lattice and $\perp \in L$ any element. Then for all $x, y \in L$, $x \wedge \perp = y \wedge \perp$ and $x \vee \perp = y \vee \perp$ together imply $x = y$.*

Corollary 2.2. *For any $\perp \in L$ the map $x \mapsto \langle x \wedge \perp, x \vee \perp \rangle$ is injective.*

This allows us to reason about elements of L as pairs $\langle x \wedge \perp, x \vee \perp \rangle$. We define the corresponding injective map which splits an element into its pair of components as

$$\delta : L \to L \times L, \quad x \mapsto \langle x \wedge \perp, x \vee \perp \rangle. \tag{2.16}$$

Observe that for all $x \in L$ we have

$$x \sqcap f = x \wedge \perp, \; x \sqcap t = x \vee \perp$$

whence $\downarrow f \times \downarrow t$ is the product of the lower and upper sets of $\perp$ in the logical order. Hence the map δ can be described as $x \mapsto \langle x \sqcap f, x \sqcap t \rangle$ and taking values in $\downarrow f \times \downarrow t$.

Using the expanded version of the semilattice operation $\sqcap$ immediately yields that on $\downarrow f$ the information order agrees with the logical order, and dually on $\downarrow t$ the information order is the dual of the logical order. With this it is easy to see that a pair in $\downarrow f \times \downarrow t$ is always *disjoint* in the information order, meaning the information meet is the least element $\perp$.

Lemma 2.3. *The lower set of any $\top \in L$ in the semilattice order $\sqsubseteq$ induced by $\perp$ is a (bounded) distributive lattice.*

Proof. Suppose $x \sqsubseteq \top$. Expanding the definition we get

$$\begin{aligned} x = x \sqcap \top &= (x \vee \perp) \wedge (\top \vee \perp) \wedge (x \vee \top) \\ &= (x \wedge (\perp \vee \top)) \vee (\perp \wedge \top), \\ x = x \sqcap \top &= (x \wedge \perp) \vee (\top \wedge \perp) \vee (x \wedge \top) \\ &= (x \vee (\perp \wedge \top)) \wedge (\perp \vee \top). \end{aligned}$$

Thus it is easily seen that $x \wedge (\bot \vee \top) = x$ and $x \vee (\bot \wedge \top) = x$. This shows that that the lower set of $\top$ in the induced semilattice order is the interval $[(\bot \wedge \top), (\bot \vee \top)]$ in the lattice order. The converse is trivially true: The pair $\langle \bot, \top \rangle$ is complemented in the interval $[(\bot \wedge \top), (\bot \vee \top)]$, whence by the 90-degree-lemma 1.1 the order $\sqsubseteq$ induced by $\bot$ gives a bounded distributive lattice structure with greatest element $\top$. $\square$

The above lemma tells us in particular that below any element in the semilattice we can form binary joins. We denote the set of all pairs in $\downarrow f \times \downarrow t$ which do have an information join by P.

Lemma 2.4. *Let $\sqcap$ be induced by distributive lattice operations $\wedge$ and $\vee$ as in (1.12). Then each $x \in (L, \sqcap, \bot)$ is the least upper bound of the disjoint pair $\delta(x) = \langle x \sqcap t, x \sqcap f \rangle$.*

Proof. Certainly both $x \sqcap t$ and $x \sqcap f$ are below x in the semilattice order $\sqsubseteq$. Therefore by Lemma 2.3 we can use x to calculate their join:

$$\begin{aligned}
(x \sqcap t) \sqcup (x \sqcap f) &= (x \vee \bot) \sqcup (x \wedge \bot) \\
&= ((x \vee \bot) \wedge x) \vee ((x \wedge \bot) \wedge x) \vee ((x \vee \bot) \wedge (x \wedge \bot)) \\
&= x \vee (x \wedge \bot) \vee (x \wedge \bot) \\
&= x \vee (x \wedge \bot) = x
\end{aligned}$$

$\square$

In the next subsection we will prove that Lemma 2.3 together with Lemma 2.4 implies that δ is an order-embedding which preserves meets and existing joins.[3] We use this fact in the proof of the following proposition.

Proposition 2.5. *For a semilattice operation $\sqcap$ induced by an element $\bot$ of a bounded distributive lattice $(L, \wedge, \vee, f, t)$ the following holds:*

(i) For all triples x, y, z with $x, y \in \downarrow t$ and $z \in \downarrow f$, the existence of $x \sqcup z$ and $y \sqcup z$ implies the existence of $x \sqcup y \sqcup z$.

(ii) For all triples x, y, z with $x \in \downarrow t$ and $y, z \in \downarrow f$, the existence of $x \sqcup y$ and $x \sqcup z$ implies the existence of $x \sqcup y \sqcup z$.

Proof. Note that (i) is equivalent to the statement (ii) for the opposite lattice order $(L, \geq)$, whence we prove only (i). Suppose $x, y \in \downarrow t$ and $z \in \downarrow f$ and the joins $x \sqcup z$ and $y \sqcup z$ exist. That means $\langle z, x \rangle$ and $\langle z, y \rangle$ are elements of P. Since in our setting the axioms ($\Diamond$) and ($\angle$) as described below hold, we know by Lemma 2.6 that the map δ is an order-embedding and $\sqcup$ its inverse, and furthermore both maps preserve finite meets and existing joins. Now consider $(x \sqcup z) \vee (y \sqcup z)$.

[3] In fact preservation of meets and existing joins is independent of the order-embedding and only requires distributivity of lower sets of points.

Since $\delta \circ \sqcup$ is the identity on P we know that $(x \sqcup z) \sqcap t = (x \sqcup z) \vee \bot = x$ and $(x \sqcup z) \sqcap f = (x \sqcup z) \wedge \bot = z$ and similarly for $y \sqcup z$. We calculate $\delta((x \sqcup z) \vee (y \sqcup z))$.

$$\begin{aligned}
((x \sqcup z) \vee (y \sqcup z)) \sqcap t &= ((x \sqcup z) \vee (y \sqcup z)) \vee \bot \\
&= ((x \sqcup z) \vee \bot) \vee ((y \sqcup z) \vee \bot) \\
&= (x \vee y) \\
&\overset{x,y \in \downarrow t}{=} x \sqcup y \\
((x \sqcup z) \vee (y \sqcup z)) \sqcap f &= ((x \sqcup z) \vee (y \sqcup z)) \wedge \bot \\
&= ((x \sqcup z) \wedge \bot) \vee ((y \sqcup z) \wedge \bot) \\
&= z \vee z = z.
\end{aligned}$$

Thus $\delta((x \sqcup z) \vee (y \sqcup z)) = \langle z, x \sqcup y\rangle = \langle z, x\rangle \sqcup \langle z, y\rangle$ and since $\sqcup$ preserves existing joins we thereby have $(x \sqcup z) \vee (y \sqcup z) = (x \sqcup z) \sqcup (y \sqcup z) = x \sqcup y \sqcup z$. $\square$

We summarise: The semilattice $(L, \sqcap, \bot)$ induced by an element $\bot$ of a bounded distributive lattice $(L, \wedge, \vee, f, t)$ satisfies the following axioms.

- ($\Diamond$) *bounded distributivity.* The lower set of any element is a bounded distributive lattice in the induced order.
- ($\angle$) *orthogonality*[4]. The pair $\langle f, t\rangle$ is disjoint and the map $x \mapsto (x \sqcap f) \sqcup (x \sqcap t)$ is the identity.
- (⧉) *cube completeness.* For all triples x, y, z with $x, y \in\downarrow t$ and $z \in\downarrow f$, the existence of $x \sqcup z$ and $y \sqcup z$ implies the existence of $x \sqcup y \sqcup z$. A similar statement for the existence of joins holds for $x \in\downarrow t$ and $y, z \in\downarrow f$.

2.2 From the semilattice to the lattice

In this subsection we show that a semilattice which satisfies the axioms ($\Diamond$) and (⧉) supports a distributive lattice structure on a subset, and this subset equals the whole lattice if also ($\angle$) is satisfied.

Suppose $(L, \sqcap, \bot)$ is a meet-semilattice with bottom element and $\langle f, t\rangle$ is a disjoint pair, that is $f \sqcap t = \bot$. Further assume that L satisfies the bounded distributivity axiom ($\Diamond$). As in the previous subsection we make the following definitions:

$$\begin{aligned}
&P := \{\langle x, y\rangle \in\downarrow f \times \downarrow t \mid x \sqcup y \text{ exists}\} \\
&\delta : L \to P, \quad x \mapsto \langle x \sqcap f, x \sqcap t\rangle \\
&\phi : L \to L, \quad x \mapsto (x \sqcap f) \sqcup (x \sqcap t)
\end{aligned}$$

Equipped with the product order, P is a meet-semilattice with least element $\langle \bot, \bot\rangle$. Obviously ϕ is below the identity on L. It is clear that δ preserves

[4]The name *orthogonality* is chosen because of the axiom's similarity to an orthogonal decompostion of a Hilbert space.

binary meets:

$$\begin{aligned}\delta(x \sqcap y) &= \langle (x \sqcap y) \sqcap f, (x \sqcap y) \sqcap t\rangle \\ &= \langle (x \sqcap f) \sqcap (y \sqcap f), (x \sqcap t) \sqcap (y \sqcap t)\rangle \\ &= \delta(x) \sqcap \delta(y).\end{aligned}$$

Also observe that P is the maximal subset of $\downarrow f \times \downarrow t$ where $\sqcup$ is defined, and ϕ is the composition $\sqcup \circ \delta$.

Suppose $\langle x, y\rangle \in P$, i.e. $x \sqcup y$ exists. Then, using the distributive law in $\downarrow(x \sqcup y)$ one finds

$$\begin{aligned}(x \sqcup y) \sqcap f &= (x \sqcup y) \sqcap (f \sqcap (x \sqcup y)) \\ &= (x \sqcap (x \sqcup y) \sqcap f) \sqcup (y \sqcap (x \sqcup y) \sqcap f) \\ &= (x \sqcap f) \sqcup (y \sqcap f) \\ &= x \sqcup \bot = x\end{aligned}$$

and similarly for $(x \sqcup y) \sqcap t$. Thus $\delta \circ \sqcup = \mathrm{id}_P$. Together with $\sqcup \circ \delta \sqsubseteq \mathrm{id}_L$ this gives an adjunction $L \underset{\sqcup}{\overset{\delta}{\rightleftarrows}} P$. Then δ as the left adjoint preserves all existing joins and ϕ is idempotent by the triangular identity of the adjunction. Furthermore the image $\sqcup(P)$ is the set of fixed points of ϕ and by $\delta \circ \sqcup = \mathrm{id}_P$ the map δ restricted to $\sqcup(P) = Fix\phi$ is a bijection. To summarise, δ is a surjective semilattice homomorphism which also preserves all existing joins and in case L satisfies the axiom $(\angle)$ we additionally have $Fix\phi = L$ and δ is an isomorphism. Dually $\sqcup$ as a right adjoint preserves all existing meets, and by definition also all existing joins. If $(\angle)$ holds then $\sqcup$ is the inverse of δ. We note this result:

Lemma 2.6. *If $(L, \sqcap, \bot)$ satisfies $(\Diamond)$ and $(\angle)$ –in particular if $\sqcap$ comes from a bounded distributive lattice by (1.12)– then $\delta : L \to P$ is an order-embedding which preserves binary meets and all existing joins.*

Proof. We have seen above that under the hypothesis of this lemma δ is an isomorphism of meet-semilattices which also preserves all existing joins, since it is a left adjoint. For the order-embedding suppose $\delta(x) \sqsubseteq \delta(y)$. Since $\sqcup$ is monotone and in this setting also the inverse of δ we get $x = \sqcup(\delta(x)) \sqsubseteq \sqcup(\delta(y)) = y$. □

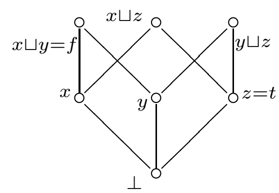
Now we are ready to define new lattice operations on P which make f into the smallest and t into the greatest element. We embed f and t into P as $\delta(f) = \langle f, \bot\rangle$ and $\delta(t) = \langle \bot, t\rangle$.

$$\langle x, x'\rangle \vee \langle y, y'\rangle := \langle x \sqcap y, x' \sqcup y'\rangle \quad (2.17)$$

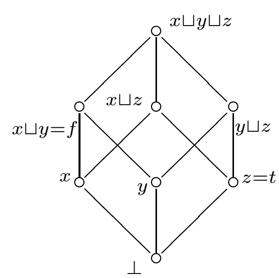
$$\langle x, x'\rangle \wedge \langle y, y'\rangle := \langle x \sqcup y, x' \sqcap y'\rangle \quad (2.18)$$

Remark. These operations can be defined on all of $\downarrow f \times \downarrow t$ which is a bounded distributive lattice under $\sqsubseteq$ with smallest element $\langle \bot, \bot\rangle$ and greatest element $\langle f, t\rangle$. However, in general $\downarrow f \times \downarrow t$ will have more elements than L, so we would obtain some kind of completion of L where for example $\bot$ has a complement. As we have seen working with $P \cong Fix\phi$ in the absence of ($\angle$) gives an underestimation of L.

The next goal is to show that P is closed under the operations $\vee$ and $\wedge$. This is the case precisely when for $\langle x, x' \rangle$ and $\langle y, y' \rangle$ in P the pairs $\langle x \sqcap y, x' \sqcup y' \rangle$ and $\langle x \sqcup y, x' \sqcap y' \rangle$ have a join in L. Since these two existence problems are dual by exchanging the roles of f and t we concentrate on finding the join of $x \sqcup y$ and $x' \sqcap y'$. Observe that if $\langle x, x' \rangle$ and $\langle y, y' \rangle$ are in P then also $\langle x, x' \sqcap y' \rangle$ and $\langle y, x' \sqcap y' \rangle$ are in P. Furthermore it is clear that the join of $x \sqcup y$ and $(x' \sqcap y') \sqcap (x' \sqcap y')$, if it exists, must solve our original problem. Thus our original existence problem can be reduced to a situation where we have pairs $\langle x, z \rangle$ and $\langle y, z \rangle$. This is where our axiom (⧉) comes in. It ensures the existence of $x \sqcup y \sqcup z$. To illustrate that the axiom (⧉) cannot be derived from ($\Diamond$) and ($\angle$) consider the following example:



In this semilattice the axioms ($\Diamond$) and ($\angle$) are satisfied and $\langle x, z \rangle$ and $\langle y, z \rangle$ are in P but $x \sqcup y \sqcup z$ does not exist. One can also see where the name of the axiom *cube completeness* comes from: The join $x \sqcup y \sqcup z$ completes the diagram to a cube.



We continue by examining the new lattice operations 2.17 and 2.18 more closely. In the following we assume that L satisfies the axioms (⧉) and ($\angle$), in which case $L \cong P$ by the isomorphisms δ and $\sqcup$. As simple calculation shows that

$$\begin{aligned}
\langle x, x' \rangle \wedge \langle y, y' \rangle &= (\langle x, x' \rangle \sqcap \langle f, \bot \rangle) \sqcup (\langle y, y' \rangle \sqcap \langle f, \bot \rangle) \sqcup (\langle x, x' \rangle \sqcap \langle y, y' \rangle) \\
&= (\langle x, x' \rangle \sqcup \langle f, \bot \rangle) \sqcap (\langle y, y' \rangle \sqcup \langle f, \bot \rangle) \sqcap (\langle x, x' \rangle \sqcup \langle y, y' \rangle), \\
\langle x, x' \rangle \vee \langle y, y' \rangle &= (\langle x, x' \rangle \sqcap \langle \bot, t \rangle) \sqcup (\langle y, y' \rangle \sqcap \langle \bot, t \rangle) \sqcup (\langle x, x' \rangle \sqcap \langle y, y' \rangle) \\
&= (\langle x, x' \rangle \sqcup \langle \bot, t \rangle) \sqcap (\langle y, y' \rangle \sqcup \langle \bot, t \rangle) \sqcap (\langle x, x' \rangle \sqcup \langle y, y' \rangle)
\end{aligned}$$

Translating this from P to L using the isomorphism $\sqcup$ yields the familiar and more readable form

$$
\begin{aligned}
x \wedge y &= (x \sqcap f) \sqcup (y \sqcap f) \sqcup (x \sqcap y) \\
&= (x \sqcup f) \sqcap (y \sqcup f) \sqcap (x \sqcup y) \\
x \vee y &= (x \sqcap t) \sqcup (y \sqcap t) \sqcup (x \sqcap y) \\
&= (x \sqcup t) \sqcap (y \sqcup t) \sqcap (x \sqcup y)
\end{aligned}
$$

The axiom (⧉) ensures that that the used joins exist. By use of the isomorphisms δ and $\sqcup$ it is also straightforward to show that the operations $\wedge, \vee, \sqcap$ and where existent $\sqcup$ distribute over each other. Finally, if the semilattice operation $\sqcap$ was defined in a distributive lattice $(L, \wedge, \vee, f, t)$ via (1.12) then we know that $\delta(x) = \langle x \wedge \bot, x \vee \bot \rangle$ and on $\downarrow f$ the operations are $\sqcap = \vee$, $\sqcup = \wedge$ and on $\downarrow t$ the operations are $\sqcap = \wedge$, $\sqcup = \vee$. For pairs let $\wedge$ denote the new operation (2.18) and inside pairs let $\wedge, \vee$ denote the lattice operations.

$$
\begin{aligned}
\delta(x) \wedge \delta(y) &= \langle (x \wedge \bot) \sqcup (y \wedge \bot), (x \vee \bot) \sqcap (y \vee \bot) \rangle \\
&= \langle (x \wedge \bot) \wedge (y \wedge \bot), (x \vee \bot) \wedge (y \vee \bot) \rangle \\
&= \langle (x \wedge y) \wedge \bot, (x \wedge y) \vee \bot \rangle \\
&= \delta(x \wedge y)
\end{aligned}
$$

and similarly for $\delta(x) \vee \delta(y)$. Suppose we start with a semilattice $(L, \sqcap, \bot)$ which satisfies the axioms $(\Diamond)$, $(\angle)$ and (⧉), and we have defined lattice operations $\vee$ and $\wedge$ on L as in (2.17) and (2.18). Using this structure (and abusing notation again) we define a new operation $\sqcap$ on $P \cong L$ by

$$
\langle x, x' \rangle \sqcap \langle y, y' \rangle = (\langle x, x' \rangle \wedge \langle \bot, \bot \rangle) \vee (\langle y, y' \rangle \wedge \langle \bot, \bot \rangle) \vee (\langle x, x' \rangle \wedge \langle y, y' \rangle). \tag{2.19}
$$

Expanding the definitions of $\vee$ and $\wedge$ one finds $\langle x, x' \rangle \sqcap \langle y, y' \rangle = \langle x \sqcap y, x' \sqcap y' \rangle$. Thus the new operation $\sqcap$ agrees with the original semilattice operation. We summarise this section with the following theorem.

Theorem 2.7. *(i) Let $(L, \wedge, \vee, f, t)$ be a bounded distributive lattice and $\bot$ an element of L. Define an operation $\sqcap$ by*

$$
x \sqcap y = (x \wedge \bot) \vee (y \wedge \bot) \vee (x \wedge y). \tag{2.20}
$$

Then $f \sqcap t = \bot$ and the meet-semilattice $(L, \sqcap, \bot)$ satisfies the axioms $(\Diamond)$, $(\angle)$ and (⧉).

(ii) Let $(L, \sqcap, \bot)$ be a meet-semilattice with disjoint pair $\langle f, t \rangle$ such that the axioms $(\Diamond)$, $(\angle)$ and (⧉) are satisfied. Then L supports a bounded distributive lattice structure in which f is the least and t is the greatest element, and the semilattice operation and the new lattice operations distribute over each other.

(iii) If a semilattice operation $\sqcap$ is given on a bounded distributive lattice L as in (2.20) then construction (ii) gives back the original lattice operations.

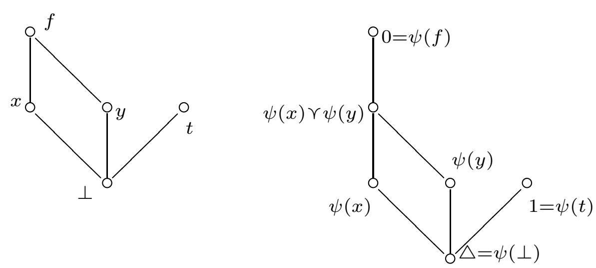
(iv) If a bounded distributive lattice structure is given on a meet-semilattice by (2.17) and (2.18), then by (2.19) one obtains the original semilattice operation.

3 Speaking categorically

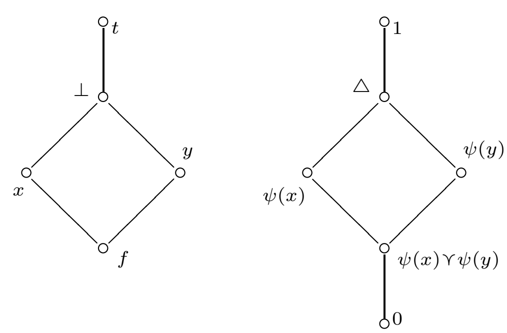
We have seen that a bounded distributive lattice together with a distinguished element is the same as a meet-semilattice with two distinguished elements that satisfies the three axioms ($\Diamond$), ($\angle$) and (⧉). In this section we examine which morphisms are preserved by the translation described in section 2, thus exhibiting an isomorphism of categories.

3.1 Morphisms of pointed semilattices

We start with an example which shows that the naive approach, which is to allow arbitrary semilattice morphisms which preserve all distinguished elements, does not suffice. Throughout this subsection we will let $(L, \sqcap, \bot, f, t)$ be a meet-semilattice with bottom element and $\langle f, t \rangle$ the distinguished pair of elements such that the three axioms ($\Diamond$), ($\angle$) and (⧉) hold, and similarly $(M, \curlywedge, \triangle, 0, 1)$ be another such structure. We call such structures *pointed meet-semilattices* and denote the class of all such semilattices by psLat. We denote the induced distributive lattices on the two pointed meet-semilattices by $(L, \wedge, \vee, f, t)$ and $(M, \cap, \cup, 0, 1)$. Suppose $\psi : L \to M$ is a semilattice homomorphism which in addition satisfies $\psi(f) = 0$ and $\psi(t) = 1$. Consider the following example:



The corresponding lattices are



Now it is easy to see that ψ is not a homomorphism of distributive lattices because ψ does not preserve the lattice meet of x and y. The problem is resolved by requiring ψ to preserve all existing joins of the semilattice. As the following proposition shows, a slightly weaker assumption can be made.

Proposition 3.1. *Let $\psi : L \to M$ be a semilattice homomorphism between pointed meet-semilattices $(L, \sqcap, \perp, f, t)$ and $(M, \curlywedge, \triangle, 0, 1)$. If ψ restricted to the lower sets $\downarrow f$ and $\downarrow t$ is a homomorphism of bounded distributive lattices such that ψ restricts to bounded distributive lattice homomorphisms $\downarrow f \to \downarrow 0$ and $\downarrow t \to \downarrow 1$ then ϕ is a homomorphism of the induced bounded distributive lattices $(L, \wedge, \vee, f, t)$ and $(M, \cap, \cup, 0, 1)$.*

Proof. Observe that by hypothesis ψ restricted to $\downarrow f$ or $\downarrow t$ preserves finite joins, and in particular $\psi(f) = 0$ and $\psi(t) = 1$. First we show that this implies that ψ preserves all existing binary joins. Suppose $x, y \in L$ and $x \sqcup y$ exists. Then $\psi(x \sqcup y)$ is an upper bound for $\{\psi(x), \psi(y)\}$ and by the axiom $(\lozenge)$ for M the join $\psi(x) \curlyvee \psi(y)$ exists. We prove $\psi(x \sqcup y) = \psi(x) \curlyvee \psi(y)$ by showing that the images under $\delta_M : z \mapsto \langle z \curlywedge 0, z \curlywedge 1\rangle$ agree. Recall that this map preserves existing joins.

$$\begin{aligned}
\psi(x \sqcup y) \curlywedge 0 &= \psi(x \sqcup y) \curlywedge \psi(f) \\
&= \psi((x \sqcup y) \sqcap f) \\
&= \psi((x \sqcap f) \sqcup (y \sqcap f)). \\
(\psi(x) \curlyvee \psi(y)) \curlywedge 0 &= (\psi(x) \curlywedge 0) \curlyvee (\psi(y) \curlywedge 0) \\
&= (\psi(x) \curlywedge \psi(f)) \curlyvee (\psi(y) \curlywedge \psi(f)) \\
&= \psi(x \sqcap f) \curlyvee \psi(y \sqcap f) \\
&= \psi((x \sqcap f) \sqcup (y \sqcap f)) \\
&= \psi(x \sqcup y) \curlywedge 0
\end{aligned}$$

and similarly for the meet with 1. Now recall that δ_M is injective on the pointed meet-semilattice M. This proves that ψ preserves all binary joins. Finally, to show that ψ preserves the lattice operations $\wedge$ and $\vee$ simply expand the definitions:

$$\begin{aligned}
\psi(x \wedge y) &= \psi((x \sqcap f) \sqcup (y \sqcap f) \sqcup (x \sqcap y)) \\
&= (\psi(x) \curlywedge 0) \curlyvee (\psi(y) \curlywedge 0) \curlyvee (\psi(x) \curlywedge \psi(y)) \\
&= \psi(x) \cap \psi(y)
\end{aligned}$$

and similarly for the join $\vee$. □

We make the class psLat into a category by taking as morphisms the semilattice homomorphisms which satisfy the additional conditions of Proposition 3.1.

3.2 Morphisms of pointed lattices

We shall call a bounded distributive lattice $(L, \wedge, \vee, f, t)$ with a distinguished element $\perp$ a *pointed bounded distributive lattice* and denote the class of these structures by pbdLat.

Proposition 3.2. *Let ψ be a homomorphism between pointed bounded distributive lattices $(L, \wedge, \vee, f, t, \perp)$ and $(M, \cap, \cup, 0, 1, \triangle)$ which preserves the distinguished element, i.e. $\psi(\perp) = \triangle$. Then ψ is a homomorphism of the induced semilattices $(L, \sqcap, \perp)$ and $(M, \curlywedge, \triangle)$.*

Proof. Simply expand the definitions:

$$\begin{aligned} \psi(x \sqcap y) &= \psi((x \wedge \perp) \vee (y \wedge \perp) \vee (x \wedge y)) \\ &= (\psi(x) \cap \triangle) \cup (\psi(y) \cap \triangle) \cup (\psi(x) \cap \psi(y)) \\ &= \psi(x) \curlywedge \psi(y). \end{aligned}$$

□

We make the class pbdLat into a category by taking as morphisms those lattice homomorphisms which preserve the distinguished element. Propositions 3.1 and 3.2 yield

Theorem 3.3. *The categories* pbdLat *and* psLat *are isomorphic.*

4 Order properties of the information order

In this section we examine the properties of the pointed meet-semilattice $(L, \sqcap, \perp, f, t)$ more closely and relate them to the order properties of its corresponding pointed bounded distributive lattice. In particular we investigate under what hypotheses joins exist.

Definition 4.1. Let $(L, \sqcap, \perp, f, t)$ be a pointed meet-semilattice and $\langle a, b\rangle \in \downarrow f \times \downarrow t$. A set of the form

$$F_a := \{x \in L \mid x \sqcap f = a\} \tag{4.21}$$

is called a *slice* of L and a set of the form

$$G_b := \{x \in L \mid x \sqcap t = b\} \tag{4.22}$$

is called a *coslice* of L.

Lemma 4.1. *Let $(L, \sqcap, \perp, f, t) \in$ psLat and $a \sqsubseteq f$. Then on the slice F_a the logical order $\leq$ coincides with the information order. Furthermore F_a is closed under all existing information joins. Dually, on the coslice G_b for $b \sqsubseteq t$ the information order is dual to the logical order and G_b is closed under all existing information joins.*

Proof. By the order-embedding δ we know that for $x, y \in F_a$ we have $x \sqcap y = x$ if and only if $(x \sqcap y) \vee \perp = x \vee \perp$. In the other component we have $(x \sqcap y) \wedge \perp = a = x \wedge \perp$ always true. Expanding the operator $\sqcap$ in logical terms and using $x \wedge \perp = y \wedge \perp = a \leq \perp$ we obtain $x \sqcap y = x$ iff $(x \wedge y) \vee \perp = x \vee \perp$. Together with the trivial equation $(x \wedge y) \wedge \perp = x \wedge \perp$ we get $x \wedge y = x$ by distributivity of $(L, \leq)$. Conversely, if $x \wedge y = x$ holds then by what is stated above we immediately get $x \sqcap y = x$.

Suppose $X \subseteq F_a$ and the join $\bigsqcup X$ exists in L. This is in particular an $\sqsubseteq$-upper bound for X and hence $\bot \wedge \bigsqcup X \leq a$. Because of $a \leq \bot$ it is not hard to see that also $a \vee \bigsqcup X$ is an $\sqsubseteq$-upper bound for X which is lower in the information order. Indeed, $\bot \wedge (a \vee \bigsqcup X) = a$ and $\bot \vee (a \vee \bigsqcup X) = \bot \vee \bigsqcup X$. By minimality then $a \vee \bigsqcup X = \bigsqcup X$ and hence $\bigsqcup X \in F_a$.

One proves the claims about the set G_b in exactly the same way, by reversing the logical order. □

4.1 Bounded completeness

Lemma 4.2. *Suppose* $(L, \leq) \in \mathsf{pbdLat}$ *and* $\bot$ *is its distinguished element. Define the information order* $\sqsubseteq$ *on* L *by the semilattice operation (1.12). Then* $y \in L$ *is an upper bound for a subset* $X \subseteq L$ *in* $(L, \sqsubseteq)$ *if and only if*

$$\bot \wedge y \leq \bot \wedge \bigwedge X \text{ and } \bot \vee y \geq \bot \vee \bigvee X \tag{4.23}$$

Proof. From Proposition 2.6 we know that y is an upper bound for X in $(L, \sqsubseteq)$ if and only if

$$\forall x \in X. \bot \wedge y \leq \bot \wedge x \text{ and } \bot \vee y \geq \bot \vee x. \tag{4.24}$$

Now just observe that $\bot \wedge \bigwedge X = \bigwedge_{x \in X} \bot \wedge x$ and $\bot \vee \bigvee X = \bigvee_{x \in X} \bot \vee x$. □

Remark. If y is an upper bound for X in $(L, \sqsubseteq)$ then

$$\bot \wedge y \leq \bot \wedge \bigwedge X \leq \bigwedge X \leq \bigvee X \leq \bot \vee \bigvee X \leq \bot \vee y. \tag{4.25}$$

Corollary 4.3. *An element* y *of* L *is the join of a subset* X *in the information order induced by an element* $\bot$ *if the equations*

$$\bot \wedge y = \bot \wedge \bigwedge X \text{ and } \bot \vee y = \bot \vee \bigvee X \tag{4.26}$$

hold.

Proof. The equations (4.26) imply (4.23), so if y satisfies (4.26) then y is an upper bound for X in the information order. Further it is clear that y and X have the same upper bounds in $\sqsubseteq$, whence $y = \bigsqcup X$. □

We will show that if $(L, \leq)$ is a complete lattice then $(L, \sqsubseteq)$ is bounded complete. Suppose $X \subseteq L$ has an upper bound $\top$ in the information order $\sqsubseteq$. We define a candidate for the least upper bound:

$$\hat{x} := \left(\top \wedge \bigvee X\right) \vee \bigwedge X = \left(\top \vee \bigwedge X\right) \wedge \bigvee X \tag{4.27}$$

First we prove the equality in the above definition.

$$\begin{aligned} \left(\top \wedge \bigvee X\right) \vee \bigwedge X &= \left(\top \vee \bigwedge X\right) \wedge \left(\bigvee X \vee \bigwedge X\right) \\ &= \left(\top \vee \bigwedge X\right) \wedge \bigvee X \end{aligned}$$

These two dual definitions of $\hat{x}$ will as usual shorten the following proofs. We compute $\delta(\hat{x})$.

$$\begin{aligned}\hat{x} \wedge \bot &= \Big((\top \wedge \bot) \wedge \bigvee X\Big) \vee \Big(\bot \wedge \bigwedge X\Big) \\ &\overset{(4.25)}{=} (\top \wedge \bot) \vee \Big(\bot \wedge \bigwedge X\Big) \\ &\overset{(4.25)}{=} \bot \wedge \bigwedge X.\end{aligned}$$

Dually one obtains

$$\hat{x} \vee \bot = \bot \vee \bigvee X.$$

By Corollary 4.3, $\hat{x}$ is the information join of X. We summarise

Theorem 4.4. *If $(L, \leq)$ is a complete distributive lattice then $(L, \sqcap, \bot)$ is bounded complete.*

Note that although the definition of $\hat{x}$ depends on the chosen upper bound $\top$, all terms involving $\top$ eventually drop out of our calculations and the join of X in the information order is independent of $\top$.

Corollary 4.5. *If $(L, \leq)$ is a complete lattice then $(L, \sqcap, \bot)$ has non-empty meets.*

Proof. If $X \subseteq L$ is non-empty then the set of lower bounds of X in the information order $\sqsubseteq$ is bounded above by an element of X. By Theorem 4.4 the lower bounds of X have a join. □

Also observe that in case a subset X is finite and bounded by an element $\top$, the structure of the join $\bigsqcup X$ is the same as in (1.13), whence below any element $\top$ of $(L, \sqsubseteq)$ the finitary joins and meets $\sqcup$, $\sqcap$ distribute over each other.

4.2 Directed completeness

Another type of subsets of $(L, \sqsubseteq)$ which we would like to have joins is directed subsets. If we restrict our attention to complete lattices $(L, \leq)$ then directed completeness is equivalent to $(L, \sqsubseteq)$ being a Scott domain. We start by exhibiting a class of lattices which in general fail to be directed complete in the information order.

Lemma 4.6. *If $(L, \leq)$ is a regular frame then for all $\bot \in L$ there exists a filter $F \subseteq L$ such that*

$$\forall x \in F.\, x \vee \bot = t \text{ and } \bigwedge F \leq \neg\bot$$

where $\neg\bot$ denotes the pseudocomplement of $\bot$.

Proof. Recall that a poset is called *regular* if every element is the join of elements well-inside it. In a frame the well-inside relation can be described as

$$u \lessdot \bot :\Leftrightarrow \bot \vee \neg u = t.$$

Observe that $\bot \vee \neg u = t$ implies $u \leq \bot$. Pick $\bot \in L$ and define $F = \{\neg u \in L \,|\, u \lessdot \bot\}$. Clearly all elements in F have join t with $\bot$. We show that F is a filter. Since $\neg$ is an antitone operation on L and $u' \leq u \lessdot \bot \Rightarrow u' \lessdot \bot$ we know that F is an upper set. Suppose u and u' are well-inside $\bot$. By hypothesis L is a Heyting algebra, whence $\neg(u \vee u') = \neg u \wedge \neg u'$ (see [5, chapter I]). Then

$$\neg(u \vee u') \vee \bot = (\neg u \wedge \neg u') \vee \bot = (\neg u \vee \bot) \wedge (\neg u' \vee \bot) = t \wedge t = t$$

so the elements well-inside $\bot$ have finite joins and F has finite meets. By hypothesis L is regular, so $\bot = \bigvee_{u \lessdot \bot} u$. Then

$$\begin{aligned} \bot \wedge \bigwedge F &= \left(\bigvee_{u \lessdot \bot} u\right) \wedge \bigwedge F \\ &= \bigvee_{u \lessdot \bot} \left(u \wedge \bigwedge F\right) \\ &= \bigvee_{u \lessdot \bot} f = f \end{aligned}$$

which shows that $\bigwedge F \leq \neg\bot$. □

Lemma 4.7. *Let $(L, \leq)$ be a complete lattice and $\bot$ an element of L such that there exists a filter $F \subseteq L$ with*

$$\forall x \in F.\, x \vee \bot = t \textit{ and } \bot \wedge \bigwedge F = f.$$

Then F is directed in the information order $(L, \sqsubseteq)$ defined by $\bot$ and has an upper bound if and only if $\bot$ has a complement in the logical order $\leq$.

Proof. By hypothesis F is a filter, so in particular downward directed in the order $\leq$. Recall that on $\downarrow\bot$ the information order $\sqsubseteq$ is the dual to the logical order, so the set $\{x \wedge \bot \,|\, x \in F\}$ is (upward) directed in the logical order. By hypothesis $(\bot \vee \cdot)$ is constant on F, so in particular the set $\{x \vee \bot \,|\, x \in F\}$ is directed in the information order. By the order-embedding δ then F is directed with respect to $\sqsubseteq$. Suppose that y is an upper bound for F in $(L, \sqsubseteq)$. By (4.23) then

$$\begin{aligned} y \wedge \bot &\leq \bot \wedge \bigwedge F = f, \\ y \vee \bot &\geq \bot \vee \bigvee F = t. \end{aligned}$$

Hence y is the complement of $\bot$. Conversely, if $\bot$ has a complement in the logical order then the complement is the greatest element of $(L, \sqsubseteq)$, so in particular an upper bound for F. □

Lemma 4.6 and Lemma 4.7 together yield

Proposition 4.8. *The meet-semilattice $(L, \sqcap, \bot)$ induced by an element $\bot$ of a regular frame $(L, \leq)$ is a Scott domain if and only if $\bot$ is complemented.*

Since the interesting examples of pointed meet-semilattices fail to have a top element we can conclude that in general the logical order should not be "too rich".

Theorem 4.9. *Let* $(L, \wedge, \vee, f, t, \bot)$ *be a pointed bounded distributive lattice. If* $(L, \wedge, \vee, f, t)$ *is complete and*

1. *every slice* F_a *as defined in (4.21) is closed under all existing logical joins*
2. *every coslice* G_b *as defined in (4.22) is closed under all existing logical meets*

then the corresponding pointed meet-semilattice $(L, \sqcap, \bot)$ *is directed complete.*

Proof. Let $D \subseteq L$ be a directed subset of the meet-semilattice $(L, \sqcap, \bot)$. We show that the image of D under δ has a join.

$$\delta(D) = \{\langle x_d, y_d\rangle \in P \mid d \in D\}, \quad x_d = d \sqcap f,\, y_d = d \sqcap t.$$

For every $d \in D$ consider the set $\pi_d := \{\langle x_d, y_e\rangle \mid e \in D\}$. These pairs are elements of P because for $d, e \in D$ there exists an element $e' \in D$ with $e' \sqsupseteq x_d, y_e$. Using the notation from Lemma 4.1 this is a subset of F_{x_d}, of which we know that on it the orders $\leq$ and $\sqsubseteq$ agree. By completeness of the logical order we can form the join $\bigvee \pi_d$ which by hypothesis must be in F_{x_d}. Hence we can write $\bigsqcup \pi_d = \langle x_d, \bigsqcup_{e\in D} y_e\rangle$. We abbreviate $y := \bigsqcup_{e\in D} y_e$. Now observe that for all $d \in D$ the so-formed join is an element of the coslice G_y. Again by completeness one forms the meet $\bigwedge_{d\in D} \bigsqcup \pi_d$ which by hypothesis is an element of G_y and therefore can be written as

$$\bigsqcup_{d\in D} \langle x_d, \bigsqcup_{e\in D} y_e\rangle = \bigsqcup_{d\in D} \bigsqcup_{e\in D} \langle x_d, y_e\rangle.$$

By a standard domain theory argument the above directed join equals $\bigsqcup_{d\in D}\langle x_d, y_d\rangle$. □

5 Order properties of the logical order

In this section we examine how a given structure on the information side reflects back to the logical order. Throughout this section we assume that $(L, \wedge, \vee, f, t)$ is a pointed bounded distributive lattice and $(L, \sqcap, \bot)$ the corresponding pointed meet-semilattice. In addition we assume that $(L, \sqsubseteq)$ is directed complete. For the sake of brevity we will say that a set $D \subseteq L$ is $\leq$*-directed* if D is directed in $(L, \leq)$ and similarly $\sqsubseteq$*-directed* if D is directed in $(L, \sqsubseteq)$.

5.1 Heyting implication

We show that directed completeness of the information order suffices to show existence of a wealth of Heyting implications on the logical side.

Theorem 5.1. *Let* $(L, \sqcap, \perp) \in \mathsf{psLat}$ *directed complete. We denote intervals in the information order by* $[\cdot, \cdot]$*.*

(i) Every slice F_a *has a* $\sqsubseteq$*-largest element which is the logical Heyting implication* $\perp \Rightarrow a$*. Hence every slice is an interval* $[a, \perp \Rightarrow a]$*.*

(ii) The interval $[a, \perp \Rightarrow a]$ *is a Heyting algebra under the logical order which coincides with the information order on this set. Consequently, if* $(L, \sqcap, \perp)$ *is also bounded complete then* $[a, \perp \Rightarrow a]$ *is a frame.*

(iii) Every coslice G_b *has a* $\sqsubseteq$*-largest element which is the logical Heyting implication in the lattice* $(L, \leq)^{\mathrm{op}}$*, denoted by* $b \Leftarrow \perp$*.*

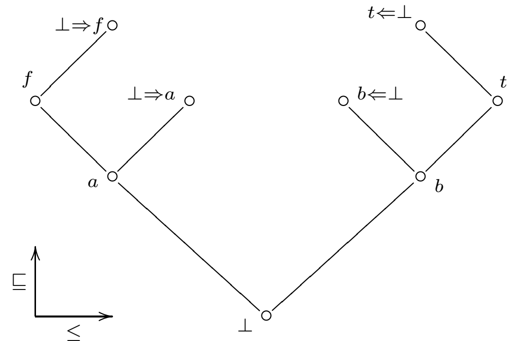
(iv) The interval $[b, b \Leftarrow \perp]$ *is a Heyting algebra in the information order which is the opposite of the logical order on this set. Consequently if* $(L, \sqcap, \perp)$ *is also bounded complete then* $[b, b \Leftarrow \perp]$ *is a frame.*

Proof. Since (*iii*) is dual to (*i*) and (*iv*) is dual to (*ii*) we prove only (*i*) and (*ii*). A proof for the other two items can be obtained by reversing the roles of f and t.

First note that F_a is $\leq$-confinal in the set $\{x \in L \mid x \wedge \perp \leq a\}$. Indeed, if $x \wedge \perp \leq a$ then $x \leq x \vee a \in F_a$. Therefore, if a logical join of F_a exists then it must be the Heyting implication $\perp \Rightarrow a$. We know that on F_a the two orders agree by Lemma 4.1. Next observe that F_a is directed. This follows immediately from the fact that F_a is closed under binary logical joins. By directed completeness of $(L, \sqsubseteq)$ then $\bigsqcup F_a$ exists. Then Lemma 4.1 states that this join is an element of F_a. This completes the proof of (*i*).

Now suppose $b, c \in F_a$. We want to show the existence of $\bigvee \{x \in F_a \mid x \wedge b \leq c\}$. Note that this is in general not the Heyting implication $b \Rightarrow c$ in all of L, whence we use the notation $(b \Rightarrow c)_a$. Similar to the proof of (*i*) first observe that the set $D = \{x \in F_a \mid x \wedge b = c \wedge b\}$ is cofinal in $\{x \in F_a \mid x \wedge b \leq c\}$, simply by joining each element x with $c \wedge b$, which preserves the meet with $\perp$ because both b and c are elements of F_a. Now it is clear by the distributive law that D is closed under binary logical joins and thus D is $\sqsubseteq$-directed. With Lemma 4.1 one shows that $\bigsqcup D$ must be an element of F_a. Thus the Heyting implication $(b \Rightarrow c)_a$ exists and $F_a = [a, \perp \Rightarrow a]$ is a Heyting algebra. If in addition $(L, \sqsubseteq)$ is bounded complete then $[a, \perp \Rightarrow a]$ is a frame. This completes the proof of (*ii*). □

We visualise the intervals by the following diagram.



Remark. In case $a = \bot$ the Heyting implication $(b \Rightarrow c)_\bot$ is indeed the Heyting implication in all of L, which is due to the fact that $F_\bot = [\bot, f]$.

Remark. If $(L, \sqsubseteq)$ is directed complete then $\bot$ has a pseudocomplement $\bot \Rightarrow f$ and a co-pseudocomplement $t \Leftarrow \bot$ which is the $\leq$-smallest element x such that $\bot \vee x = t$.

Corollary 5.2. *If $(L, \sqsubseteq)$ is both directed complete and bounded complete then for each $x \in L$ the lower set of x in the information order is a frame.*

Proof. Under the hypothesis we know by Theorem 5.1 that both $[\bot, f]$ and $[\bot, t]$ are frames, whence the image $\delta(\downarrow x) = [\bot, x \sqcap f] \times [\bot, x \sqcap t]$ is the product of two frames and thereby a frame. Now recall that δ is an isomorphism which preserves finite meets and all joins, and so does its inverse $\sqcup$. □

5.2 Completeness

Theorem 4.4 says that completeness of the logical order yields bounded completeness of the information order. Now we show that under the additional assumption of directed completeness the converse of Theorem 4.4 also holds.

Theorem 5.3. *If $(L, \sqcap, \bot)$ is bounded complete and directed complete then the corresponding logical order on L is complete.*

Proof. The axiom (⧉) assures that the set $\delta(L) = P \subseteq [\bot, f] \times [\bot, t]$ is closed under the operation

$$(\langle x, x' \rangle, \langle y, y' \rangle) \mapsto \langle x \sqcap x', y \sqcup y' \rangle$$

which yields the logical binary join. We extend this operation to an infinitary operation in a straightforward way.

Recall that if L under the information order is directed complete and bounded complete then $[\bot, f]$ and $[\bot, t]$ are frames. We therefore may form

$$m := \bigsqcap_{x \in X} x \sqcap f \text{ and } j := \bigsqcup_{x \in X}^{[\bot, t]} x \sqcap t.$$

Here $\bigsqcup^{[\bot,t]}$ indicates that we compute the join in $[\bot, t]$. Our problem is that the pair $\langle m, j\rangle$ may not be in P and therefore not represent an element of L. First observe that since P is a lower set in $[\bot, f] \times [\bot, t]$ we have $\langle m, x \sqcap t\rangle \in P$ for every $x \in X$. With the cube completeness axiom one can inductively show that for all finite $X_f \subseteq X$ the pair $\left\langle m, \bigsqcup_{x\in X_f} x \sqcap t\right\rangle$ is in P. Clearly the set

$$\left\{\left\langle m, \bigsqcup_{x\in X_f} x \sqcap t\right\rangle \;\middle|\; X_f \subseteq X \text{ finite}\right\}$$

is $\sqsubseteq$-directed in P and by directed completeness therefore has a join in P. By Lemma 4.1 the join has first component m. Showing that the second component is actually j is a standard domain theory argument and just relies on the fact that $[\bot, t]$ is a join-semilattice. Thus we have an element $\langle m, j\rangle \in P$ such that

$$m = \bigvee_{x\in X} x \wedge \bot, \quad j = \bigvee_{x\in X} x \vee \bot. \tag{5.28}$$

Finally, note that δ is not only an order-embedding in the information order, but also in the logical order. Indeed, monotonicity is trivial, and if $\delta(x) \leq \delta(y)$ then

$$\begin{aligned} (x \wedge \bot) \vee (y \wedge \bot) = (x \vee y) \wedge \bot &= y \wedge \bot \\ (x \vee \bot) \vee (y \vee \bot) = (x \vee y) \vee \bot &= y \vee \bot \end{aligned}$$

which by distributivity implies $x \vee y = y$. Hence (5.28) yields $m \sqcup j = \bigvee X$. □

Remark. Although logical meets can be constructed from logical joins, it is worth noting that the meets have a similar presentation in terms of the information order. From the construction of the logical join in the preceding proof one can see that

$$\begin{aligned} (\forall x \in X.\, f \sqcap x = a) &\Rightarrow f \sqcap \bigvee X = a \\ (\forall x \in X.\, t \sqcap x = b) &\Rightarrow t \sqcap \bigwedge X = b. \end{aligned}$$

We summarise the infinitary operations we have dealt with so far:

Theorem 5.4. *Let $(L, \sqcap, \bot, f, t) \in$ psLat and $(L, \wedge, \vee, f, t, \bot)$ its corresponding object in* pbdLat.

1. *$(L, \sqcap, \bot)$ is directed complete and bounded complete if and only if $(L, \wedge, \vee, f, t)$ is complete and all slices are closed under all logical joins and all coslices are closed under all logical meets.*

2. *Given directed completeness of $(L, \sqcap, \bot)$, for any $x \in L$ the logical Heyting implication $\bot \Rightarrow x$ and the dual Heyting implication $x \Leftarrow \bot$ exist. For all pairs $\langle a, b\rangle$ in $[\bot, f] \times [\bot, t]$ the intervals $[a, \bot \Rightarrow a]$ and $[b, b \Leftarrow \bot]$ are Heyting algebras in the information order.*

3. *If* $(L, \sqcap, \perp)$ *is directed complete and bounded complete then the lower set of any point in the information order is a frame.*

4. *If* $(L, \sqcap, \perp)$ *is directed complete and bounded complete then the logical join and meet is given by*

$$\bigvee X = \left(\bigsqcap_{x \in X} x \sqcap f\right) \sqcup \left(\bigsqcup_{x \in X}^{[\perp,t]} x \sqcap t\right) \tag{5.29}$$

$$\bigwedge X = \left(\bigsqcup_{x \in X}^{[\perp,f]} x \sqcap f\right) \sqcup \left(\bigsqcap_{x \in X} x \sqcap t\right) \tag{5.30}$$

5. *If* $(L, \wedge, \vee, f, t)$ *is complete, all slices are closed under logical joins and all coslices are closed under logical meets, then directed information joins are given as*

$$\bigsqcup D = \bigwedge_{d \in D} \bigvee_{e \in D} (d \sqcap f) \sqcup (e \sqcap t). \tag{5.31}$$

If a set $X \subseteq L$ *is bounded by* $\top$ *in the information order then the bounded join is given as*

$$\bigsqcup X = \left(\top \wedge \bigvee X\right) \vee \bigwedge X \tag{5.32}$$

6 Another isomorphism of categories

Using the results of theorem 5.4 one is lead to reason about an isomorphism of categories in the style of Theorem 3.3. From (5.29) and (5.30) one can see that it is not enough for a morphism between semilattices to preserve the existing joins, because the formulae make use of meets as well. It turns out that an assumption similar to Proposition 3.1 is to be made.

Definition 6.1. A pointed meet-semilattice which is directed complete and bounded complete is called a *pointed Scott domain.* The class of all such structures is denoted by pScottDom.

A pointed bounded distributive lattice which is complete and slices are closed under joins and coslices are closed under meets is called a *pointed complete distributive lattice.* The class of all such structures is denoted by pcdLat.

Proposition 6.1. *Let* ψ *be a morphism between pointed Scott domains* $(L, \sqcap, \perp, f, t)$ *and* $(M, \curlywedge, \triangle, 0, 1)$ *which in addition restricts to morphisms of the complete lattices* $\downarrow f \rightarrow \downarrow 0$ *and* $\downarrow t \rightarrow \downarrow 1$ *preserving all joins and meets on these subsets of* L*. Then* ψ *is a morphism of the induced pointed complete distributive lattices which preserves all joins and meets.*

Conversely, if ψ *is a morphism between pointed complete distributive lattices* $(L, \wedge, \vee, f, t, \perp)$ *and* $(M, \cap, \cup, 0, 1, \triangle)$ *which preserves all joins and meets and also the distinguished element then* ψ *is a morphism of the induced pointed Scott domains which preserves finite meets, directed joins and bounded joins.*

Proof. All we need for the proof follows from the translations of the infinitary operations (5.29) – (5.32) and the observation that if ψ restricted to $\downarrow f$ or $\downarrow t$ preserves all joins and meets in particular the distinguished pair and the bottom element are preserved. $\square$

Corollary 6.2. *The categories* pScottDom *and* pcdLat *are isomorphic.*

6.1 Examples

We conclude this paper with a class of examples which illustrate that the hypotheses in Theorem 4.9 are necessary and exhibits the relationship of this work to [6].

Lemma 6.3. *Let $(S, \sqcap, \sqcup, 0, 1)$ be a bounded distributive lattice and L_-, $L_+ \subseteq S$ be sublattices such that $\{0, 1\}$ is contained in both of them. Define $L = \{\langle x, y\rangle \in L_- \times L_+ \mid x \sqcap y = 0\}$. Then L with the product order is a pointed meet-semilattice with distinguished pair $f = \langle 1, 0\rangle$ and $t = \langle 0, 1\rangle$.*

Proof. First observe that the bottom element of L is $\perp = \langle 0, 0\rangle$. Since the binary meet $\sqcap$ in S preserves meets and joins in its arguments, we get that L is closed under binary meets and the lower set of an element $\langle x, y\rangle$ of L is a bounded distributive lattice. Hence L satisfies the axiom ($\Diamond$). Further by the distributive law $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcup z)$ it is easy to see that L is also cube complete. The orthogonality axiom ($\angle$) is easily checked as well. By observing that $\langle x, y\rangle \sqcap f = \langle x, 0\rangle$ and $\langle x, y\rangle \sqcap t = \langle 0, y\rangle$ it is clear that the isomorphism between L and P takes the nice form $\langle x, y\rangle \mapsto \langle\langle x, 0\rangle, \langle 0, y\rangle\rangle$ whence $\downarrow f \cong L_-$ and $\downarrow t \cong L_+$. $\square$

Lemma 6.4. *If $(S, \sqcap, \sqcup, 0, 1)$ is a frame and L_-, L_+ are subframes of S such that $\{0, 1\}$ is contained in both of them then L is a pointed Scott domain.*

Proof. Bounded completeness of L is trivial. A twofold application of the frame distributivity law yields that L is also closed under directed joins. $\square$

A generic instance of this is a bitopopogical space, that is a set S together with two topologies $\mathcal{T}_-$ and $\mathcal{T}_+$. It is crucial in this setting that the infinitary join in both subframes is the same as the infinitary join in the power set of S, as the following example illustrates.

Let $S = \mathcal{P}(X)$ be the power set of a topological space $(X, \mathcal{T})$ and let $L_- = S$ and L_+ be the closed sets of the topological space X. By Lemma 6.3 the resulting semilattice L is an element of the class psLat. The logical order on L is complete. Indeed, if $\{\langle N_i, P_i\rangle\}_{i\in I}$ is a family of points in L then we can form $\bigcap_{i\in I} N_i$ and $\bigcup_{i\in I} P_i$. the join of the P_i in L_+ is obtained by the topological closure $\overline{\bigcup_{i\in I} P_i}$. To recover disjunctness we set $\left\langle \bigcap_{i\in I} N_i \setminus \overline{\bigcup_{i\in I} P_i}, \overline{\bigcup_{i\in I} P_i} \right\rangle$ as the logical join. Now suppose $A \subseteq X$ is a non-open subset. Then the slice F_A consists of all the pairs $\langle A, C\rangle$ with C closed and disjoint from A. Since A is assumed to be non-open we know that $A \setminus \overline{\bigcup_{\langle A,C\rangle\in F_A} C}$ is strictly smaller than

A, whence the slice F_A is not closed under logical joins. The same slice serves as an example of a subset of L which is directed in the information order but does not have an information join. Therefore completeness of the logical order alone does not imply directed completeness of the information order.

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